GEOGRAPHIC AREA AND MAP PROJECTIONS*

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A basic truism of geography is that the incidence of phenomena differs from place to place on the surface of the earth. Theoretical treatises that assume a uniformly fertile plain or an even distribution of population are to this extent deficient. As Edgar Kant[1] has put it:

> "The theoretical conceptions, based on hypotheses of homogeneous distribution must be adapted to geographical reality. This implies, in practice, the introduction of corrections with regard to the existence of *blank districts, deserts of a phenomenon, massives or special points*. That is to say that in practice we have to take into especial consideration the anisotropical qualities of the *area geographica..*"

The *ceteris paribus* assumptions that are repugnant to a geographer are those which conflict seriously with the fundamental fact that the distribution of phenomena on the surface of the earth is highly variable. Von Thünen[2], for example, postulates a uniform distribution of agricultural productivity; his economic postulates are no less arbitrary, but they disturb the geographer somewhat less. Christaller's central place theory is in a similar category; for the necessary simplifying assumptions, among them a uniform distribution of purchasing power, are unsatisfactory from a geographic point of view.[3] In order to test the theory empirically, one must find rather large regions in which the assumptions obtain to a fairly close approximation. The theory can, of course, be made more realistic by relaxing the assumptions, but this generally entails an increase in complexity. An alternate approach, hopefully simpler but equivalent, is to remove the differences in geographic distribution by a modification of the geometry or of the geographic background. This has been attempted by other geographers with some success, but without clear statement of the problem.

Map projections always modify certain geometric relations and hence would seem well suited to the present task. However, instead of considering the earth to be an isotropic closed surface (as is traditional in the study of map projections), account can be taken of an uneven distribution of a phenomenon on this surface - the *area geographica*. The topic is approached by an examination of a number of published maps called cartograms in current cartographic parlance. Attention is here directed toward those types of cartograms that appear amenable to the metrical concepts of the theory of map projections, with no attempt at definition of the rather vague term cartogram.

EXAMPLES OF CARTOGRAMS

Cartograms are of many types. The accompanying illustration showing "A New Yorker's Idea of the United States of America" (Fig. 1) contains several interesting notions. The thesis of cognitive behaviorism suggests that people behave in accordance with the external environment, not as it actually is, but as they believe it to be.[4] In this vein, the cartogram presented can be considered to illustrate one type of psychological distortion of the geographic environment that may occur in the minds of many persons. It is clear that the distortion is related to distance. Furthermore, the areas of the states are not in correct proportion; Florida, for example, appears inordinately large. Hence distortion of area can be recognized, though a complete separation of the concepts of distance and area is not possible in this instance.

The second illustration is also a distorted view of the United States (Fig. 2), but the purpose of this cartogram is somewhat different. The areas of the states and cities are shown in proportion to their retail sales, rather than in proportion to the spherical surface areas enclosed by their boundaries. Harris' point is that the expendable income, not the number of square miles, is a more proper measure of the importance of an area - at least for the purposes of the location of economic activity. Harris also presents cartograms of the United States with map areas of the states in proportion to the number of tractors on farms and to the number of persons engaged in manufacturing.[5] Raisz[6] presents a cartogram with the areas of the states in proportion to their populations. Hoover[7] stresses a point of view similar to that of Harris and presents a different cartogram of the United States, with map areas of the cities and states in proportion to their populations. Weigert[8] recognizes that the importance of a country may be more directly proportional to its population than to its surface area and presents a cartogram placing the countries of the world in this perspective. Woytinsky and Woytinsky[9] make extensive use of a similar cartogram, reproduced here as Figure 3. Zimmermann[10] presents further examples—cartograms of world population and of output of steel by country.

Whether all these cartograms are to be considered maps, based on projections, is a matter of definition and, as such, is not really important. Raisz stresses the point that his rectangular statistical cartograms are not map projections. The network of latitude and longitude on the Woytinskys' population cartogram (Fig. 3) suggests a map projection but is actually spurious, as the Woytinskys themselves remark. However, since all maps contain distortion, the diagrams can be regarded as maps based on some unknown projection. Certainly the definition that considers a map projection to be an orderly arrangement of terrestrial positions on a plane sheet suffices. It also seems adequate to demonstrate that diagrams similar to these cartograms can be obtained as map projections. But what is the nature of these projections? No such map projections are given in the literature of the subject. The question is approached by a detailed examination of a simpler problem posed by Hãgerstrand.

Hägerstrand has been concerned with the study of migration. In discussing the cartographic problem, he states:

> "The mapping of migration for so long a period, giving the exchange of one single commune with the whole country in *countable* detail, cannot be made by ordinary methods. All parts of the country have through the flight of time been influenced by migration. However, different areas have been of very different importance. With the parishes bordering the migrational centre, the exchange has numbered hundreds of individuals a decade. At long distances only a few migrants or small groups are recorded. A map partly allowing a single symbol to be visible at its margin, partly giving space to die many symbols near its centre, calls for a large scale since we *wish to be able to count on the map"*.

It is desired to count symbols on the map. This is a clear statement of a common cartographic problem. The situation occurs frequently in the mapping of population, where high concentrations appear in restricted areas and smaller numbers are spread more thinly throughout the remainder of the map. Certainly every cartographer has at some time wished for a distribution of a phenomenon that did not seem to require that all the symbols overlap. One solution has been the introduction of so-called three-dimensional symbols.[12] An alternate solution is here suggested, based on the theory of map projections. Also note the distinction between the common geographic use of an equal-area map to illustrate the distribution of some phenomenon and Hãgerstrand's emphasis on the recovery of information recorded on the map.

In the problem as formulated by Hägerstrand , the exchange of migrants is known not to be distributed arbitrarily but is a function of distance from a center, the commune being studied. More commonly, differences from one area to another vary much more irregularly, as in, for example, the distribution of population throughout the world. Careful reading of Hägerstrand's statement suggests that the functional dependence is one of decreasing migratory exchange with increasing distance from the center. This can be recognized as a simple distance model often employed by geographers. In particular, the suspected function of distance can be postulated to be continuous and differentiable, strictly monotone decreasing, and independent of direction. If these postulates are accepted, the functional dependence can be shown on a graph as a continuous curve, in this instance a curve of negative slope. The curve can be considered a profile along an azimuth, and the expected incidence of migration could be shown on a map by isolines. This suggests that variants of the solution to Hãgerstrand's problem can be applied to many isoline maps. Population density, for example, is often illustrated by isolines drawn on maps, and an approach to the population cartograms is suggested. Hägerstrand's own solution is as follows[13]:

> "The problem is solved by the aid of a map-projection in which the distance from the centre shrinks proportionally to the logarithm of the real distance. (The method was suggested to the author by Prof. Edg. Kant. Maps of a similar kind are used for the treatise 'Paris et l'agglomération Parisienne' 1952.) The rule obviously cannot be applied to the shortest distances. Thus the area within a circle of one km radius has been reduced to a dot.
> The distortion in relation to the conventional map is of course considerable."

The basis for the choice of the logarithmic projection (Fig. 4) is not clearly indicated in this statement. An azimuthal projection that yields the desired result seems to have been plucked out of thin air. Working backward, however, the radial scale distortion is seen to be ρ^{-1} (where ρ is the spherical distance), and it can be inferred that the projection was obtained by taking the suspected function of distance as the radial scale distortion, as can be done for any of the distance models employed by geographers.[14] The space elimination at the origin is appropriate, for it excludes the commune being studied (which does not belong to the domain of migration). But is Hägerstrand's the most valid solution to the problem as formulated? The concept of primary concern is not distance but area. This is implicit in the statement that it is desired to be able to count symbols on the map. The suggestion is that the map should show the areas near the center at large scale and those at the periphery at small scale. Such maps would be useful in most studies of nodal regions. Hägerstrand's solution achieves this objective, as can be verified by calculation of the areal distortion, at least for areas near the center of the map. But so do the orthographic projection, the square-root projection, and many others. The azimuthal equidistant centered on the antipodal point also yields the desired solution and has been used for this purpose by Michels.[15] Kagami[16] suggests an alternate solution when faced with an almost identical problem. Charts for aircraft pilots have also been prepared using maps that have a large scale near the center and a small scale at the periphery.[17]

CARTOGRAMS AS PROJECTIONS

To clarify the situation, one should note that it is the areal scale, and not the linear scale, which is important. Furthermore, it is natural to require that the areal distortion be *exactly* the same as the expected or observed distribution. Somewhat more precisely, Hàgerstrand's problem can be generalized in the following manner. In the domain under consideration there are locations from which migration to the center originates. If we consider the beginning point of each migration to be an "event," each small region (element of area) will (or is likely to) contain a certain number of events. Hence with each proper partition of the domain is associated a number, and the area contained within the boundaries of the corresponding partition on the map is to be proportional to this number.

The similarity to the cartograms previously presented is now clearer. In each instance a set of non-negative numbers (people, dollars) has been associated with a set of bounded regions (cities, states, nations). The objective is to display the regions on a diagram in such a manner that the areas within the boundaries of the regions on the diagram are proportional to the number associated with the particular region. Harris recognizes the similarity of the concepts, for his cartogram “A Farm View of the United States”[18] is accompanied by a histogram of the number of tractors by states. On an equal-area projection, the number associated with each partition is the spherical (or ellipsoidal) surface area.

There seem to be two methods of attacking the details of these map projections. One assumes differentiability; the other is an analogue of the first but employs what might be called rule-of-thumb procedures. Each method has advantages and disadvantages. The differentiable cases display the similarity to equal-area map projections somewhat better, whereas the approximation methods are simpler to use with empirically obtained data. The differentiable cases also allow explicit solution for the pair of functions necessary to define a map projection. No attempt is made here to duplicate the specific cartograms illustrated; the purpose is only to indicate the class of projections to which they belong.

The data are somewhat difficult to manipulate when the partitions of area are large. It is therefore convenient to reduce the values associated with each portion of the domain to density form, and to think in terms of a continuous (integrable and differentiabIe) distribution that can be represented by isolines on a sphere. The details of this device are well known and can be omitted here[19]. The map area between given limits is then to be proportional to the total distribution between corresponding limits. The density distribution on the surface of a sphere is assumed to have been described by an equation. For equal-area projections the density of spherical surface area is always constant (unity), so that correct values are also obtained in this special situation. As is true of area, finite densities sum to a finite value, so that the density-preserving property of the projections to be achieved obtains both locally and in the large. The use of density values also facilitates the further objective that common boundaries between regions should again coincide on the final map.

The derivation of the cartograms under consideration as map projections follows directly from the preceding discussion. A mathematical analysis of this class of map projections is given in the Appendix. A special case, of some practical interest, is given here to illustrate the general method.

The distribution of population in an urban area can be described as a density function $D(\delta, \gamma)$ on a plane, using polar coordinates δ, γ. Horwood[20] has suggested one such distribution in which the density decreases monotonically from the center but also varies from one direction to the next (Fig. 5). The specific theoretical function taken by Horwood is such that the density is highest along symmetrically spaced radial streets (n in number) and less in the interstitial areas, which is not unrealistic and is easily described by trigonometric functions or Fourier series. The population is then given by the integral $\iint_R \delta D(\delta, \gamma)\, d\delta\, d\gamma$. To transform this to the map plane so that all map areas have identical densities, set

$$\iint_{R'} r|J|\,d\delta d\gamma = \iint_R \delta D(\delta,\gamma)\,d\delta d\gamma\ ,$$

or

$$\iint_{R'} r dr d\theta = \iint_R \delta D(\delta,\gamma)\,d\delta d\gamma\ ,$$

which is equivalent to $r\,|\,J\,| = \delta D(\delta, \gamma)$, where

$$\pm J = \frac{\partial r}{\partial \delta}\,\frac{\partial \theta}{\partial \gamma} - \frac{\partial \theta}{\partial \delta}\,\frac{\partial r}{\partial \gamma}\ .$$

For one solution, not necessarily the most appropriate but simple, stipulate that the transformation is to be azimuthal, that is, that $\theta = \gamma$. Then $\partial\theta/\partial\delta = 0$, $\partial\theta/\partial\gamma = 1$, and $J = \partial r/\partial\delta$. The equation to be solved for r is consequently $r^2 = 2\pi \int \delta D(\delta,\gamma)\, d\delta + g(\gamma)$, and the remaining details are matters of integration and root extraction. This example could be extended to a sphere or spheroid, but for an urban area there is little point in such. extension. The image of the original polar coordinates on the final map might appear as shown in Figure 5.

Although further details are in the Appendix, certain results from the mathematical analysis are worth noting here. It is easily shown that the transformations are a generalization of equal-area projections in the sense that *all* equal-area projections represent a special case. Moreover, this class of projections can be obtained by setting Tissot's measure of areal distortion equal to the given (expected, probable) density distribution. It is also apparent that there are an infinite number of solutions for any specific density. This suggests that additional conditions be applied. Of the many possible conditions, two are of particular interest. Since this class of projections is equivalent to projections with areal distortion, and since all conformal projections of a sphere distort area, it follows that a conformal projection with a specific areal distortion should yield a solution. The transformation also may be taken so that cost or time distances from the map center are correctly represented. Occasionally the assumption of continuity of a distribution is not warranted. The data are often in the form of discrete locations, as on a population dot map; or are grouped into areal units, such as census tracts; or refer to areal units rather than to infinitesimal locations, such as land values that refer to specific parcels of land. In these cases an analytic solution usually is not feasible and rule-of-thumb approximations are useful. Even in the case of continuous distributions, descriptive equations are difficult to obtain and, at present, are not available for geographic data, though theoretically possible. Approximation methods, therefore, are useful. They can also be used to demonstrate some of the different types of particular solutions available and some of the additional conditions that may be applied. The approximation methods are no less valid than the methods used in the differentiable cases and can be formalized to the same extent, but they are more akin to topological transformations than to those traditionally associated with cartography.

The only known description of the method used in the preparation of the cartograms previously mentioned is that given by Raisz[21]; the method used by others is presumably similar. The populations of the states are taken as given, and rectangles proportional to population on are drawn on a sheet of paper; adjacent rectangles are adjusted until neighbor relations and overall shape are approximately correct. This is illustrated in Figure 6. Though the example is very simple, there are still an infinite number of solutions, but some seem more appropriate than others. Preservation of the internal topology is one condition that seems desirable; this is in fact a requirement that the map (not the distribution) be continuous (a homeomorphism - neighborhoods are preserved under the mapping). Preservation of the shape of the external boundary is another condition that might be applied. Alternately, one might wish the boundary to map into a specific shape. These last two conditions are difficult to specify even in the analytic case. If one thinks in terms of a map of a part of the earth's surface, an obvious difficulty is that the immediately foregoing examples do not indicate where positions within the original areal units are to be placed within the corresponding partitions of the transformed image. Stated in another way, if locations in the original are described by latitude and longitude, where are the images of these lines in the transformed image? If the partitions represent states, the placement of cities is rather arbitrary, and so on. Here the differentiable cases display a distinct advantage. However, if a coordinate system is introduced in the original, and an assumption of uniform density within each partition (for example, states) is made, the difficulty can be circumvented by estimating lines of equal increments of density on the original. These lines then correspond to an equal-area grid system on a plane, and the converse. A similar method can be employed when the original data are given in the form of a dot map. If a partition has no entries, the map area should vanish, a collapsing of space or a many-to-one mapping. Figure 3 actually consists of several domains; otherwise, ocean areas would be eliminated (lines of latitude and/or longitude coincide), just as Greenland and Antarctica do not appear on the map. Although there is some population in the ocean areas, the amounts are so small as to be negligible. In the continuous case with zero density the transformation becomes many-to-one (a collapsing of space) for this part of the domain.

The approximation methods need not be discussed in more detail; they are fairly simple and do not reveal information that is not readily apparent from an examination of the equations given in the Appendix. More interesting, and more difficult to evaluate, are the geographic uses of maps obtained by the foregoing types of projections or transformations. These applications should also suggest the additional conditions to be applied in selection of a specific transformation from the infinite variety of particular solutions available.

GEOGRAPHIC APPLICATIONS

Obviously, the map projections obtained can be used as were the cartograms previously presented, for they were derived by consideration of such cartograms. These many applications need not be repeated. Further, any distribution plotted on a map using such a transformation shows a ratio; income symbolized on a map equalizing population density shows per capita income, and so on. The projections may likewise be useful as base maps in simulation or other studies in which data are plotted by computer.

It is also clear that any grid system which partitions the area of the plane map into units of equal size will yield a partitioning of the basic data into regions containing an equal number of elements when mapped back to the original domain. For example, states might be partitioned into electoral districts in such a manner that all districts contained an equal number of voters. The specific equal-area grids on a plane are infinite in number, so that this procedure is not really of much assistance. Equal-area grids are also difficult to define along irregular boundaries, and partitionings (electoral districts, and so on) are usually required to satisfy numerous additional conditions (coincide with city and county boundaries, and so on). To attempt to use the transformations in this manner seems politically impractical, though theoretically suggestive.

More interesting applications can perhaps be found in the theories of Von Thünen and Christaller. It is in this context that Harris and Hoover attempted to use their cartograms. Von Thünen assumes a uniform fertility of agricultural land, Christaller a uniform distribution of rural population or income, though both attempt to relax these unrealistic assumptions somewhat. If one postulates that agricultural fertility can be measured and varies from place to place—that is, that fertility can be described by a relation $F=f(\varphi, \lambda)$ and if one then applies a transformation of the type described, areas of high fertility will appear enlarged. One can then plot[23] an even yield (for example, in bushels) per unit of map area and, using the inverse transformation, return to the original domain. The even distribution of yields will now be uneven, and in fact corresponds to the distribution of fertility. This becomes more interesting if one adds the condition that cost distances from (or to, but not both) a market place appear as map distances from the center of the map and that the intensity of use (yields) decreases with cost distance. That is, on the map transformed so that all areas appear of equal fertility, returns are to be plotted as decreasing from the center of the map, as in the Von Thünen model. The inverse transformation will then display a distribution of intensity of use that takes into account fertility and cost distance from the market place. The measurement of agricultural fertility is by no means easy. Dunn[23] doubts that such measurement can be achieved, but the United States Department of Agriculture publishes detailed information with a ranked classification (measurement on an ordinal scale) of rural land based on its economic value. Cost distances are used in the preparation of the map projection as another application of the notion that the earth should perhaps not be treated as an isotropic sphere. It is necessary to take into account not only the shape of the earth but also the realities of transportation on its surface. Automobiles, trains, airplanes, and other media of transport can be considered to have the effect of modifying distance relations - measured in temporal or monetary units - in a complicated manner. It can be shown (see Appendix) that a density-preserving projection with a continuous and monotonic but otherwise arbitrary centrally symmetric distance function can be obtained. This distance function can be the empirically obtained cost - or time - distance from the market place.[24]

Just as the Von Thünen model can be applied to cities,[25] the foregoing discussion can be rephrased using "suitability for construction" instead of fertility. Many urban areas are already built up, and construction is no longer feasible; other areas are blighted and have but little appeal; some locations have high prestige value; site and topographic factors vary; and so on. Undoubtedly, measurement of these values is difficult. Requirements for different classes of land use differ, and some measure of intensity of use seems required. Land costs are biased, since they reflect accessibility and an estimate of potential returns. Nevertheless, the transformation and its inverse can be used as before. Such a transformation takes into account only two factors and is therefore of only limited assistance in explaining the totality of urban land uses. The currently available models of urban structure are not outstandingly more successful.

Christaller in his work on geographic location assumes a uniform distribution of the underlying rural population and then obtains sets of nested hexagonal service areas and a hierarchy of cities regularly spaced throughout the landscape. It has been shown how an uneven distribution may be made to appear uniformly distributed, and the pertinent question is whether Christaller's resulting pattern will now be observed. The answer is difficult, for several reasons. Given an empirical distribution of income and market areas, the transformation is to make the income densities uniform and to send the market areas into hexagons. It is not clear how this latter condition is to be specified in choosing a particular transformation from the infinite set. Christaller obtains hexagons from consideration of circular service areas, and it is known that only the stereographic projection sends all spherical circles into circles. The stereographic projection, however, will certainly not result from the density-preserving transformation in the general case. Conformal projections in general preserve circles as circles, but only locally, and would require satisfying both conditions of conformality and a specific areal distortion. For relatively small service areas conformal transformations may be suitable. The solution (if one exists) to this problem is obscure. It is possible, of course, to draw hexagons on a map of some region transformed in such a manner that densities are uniform and, by use of the inverse transformation, to examine the resulting pattern of curvilinear polygons in the original domain. There is a slight problem here of specifying an initial orientation for the hexagons and of fitting hexagons to the boundaries of the image region. The appearance of the transformed hexagons will of course differ for each transformation in the, infinite set. Nevertheless, an experiment of this nature has recently been completed by Getis, using expendable income data for the city of Tacoma.[27] Richardson's conformal transformations of hexagonal patterns are somewhat similar.[28] Some such procedure is also implied by Isard's schematic diagrams of a hypothetical landscape[29]. Conceptually, Isard's notions are correct, but the boundaries of the service areas will almost certainly not be straight lines, as they have been drawn in his illustrations. Conversely, one might use Vidale's method of partitioning a landscape into service areas,[3°] apply a transformation, and examine the images of the service areas to see whether they resemble hexagons. Such an empirical experiment does not appear difficult; one can choose simple density distributions and use the simpler and more obvious transformations. None of these methods is as satisfactory as a theoretical solution, of course, though they may shed further light on the nature of the problem. Christaller's hexagons also need not be retained. Another approach is to consider threshold populations, not hexagons. From this point of view the boundaries of service areas overlap and are somewhat indeterminate. Adding the concept of the range of a good enables one to define the region in terms of cost distances. In this instance the useful map projections are those which make cost distances from some location proportional to map distances from that location and which distribute densities evenly.

Christaller is also concerned with distances; his circular service areas are more akin to geodesic circles using a "subjectively valued time-cost distance" (sic), and his spacing of cities stipulates some distance between cities. Yet distances are not preserved by the transformations; preservation of all distances is certainly not possible if densities are to be uniformly distributed on a plane map. Clearly, then, application of the suggested transformations to theories similar to those of Von Thünen and Christaller is difficult and only partly successful, though promising and capable of improvement. The deficiencies are to a certain extent due to the inadequacies of the theories themselves; for, at present, they are neither sufficiently general nor explicitly formulated.

CONCLUSION

Valuable map projections can be obtained that do not conform to the traditional geographic emphasis on the preservation of spherical surface area but rather distort area deliberately to "eliminate" the spatial variability of a terrestrial resource endowment. In many ways these maps are more realistic than the conventional maps used by geographers and would be of value even if the earth were a disk, as some ancients believed. The important point, of course, is not that the transformations distort area but that they distribute densities uniformly. It is hoped that future textbook presentations on the subject of map projections will include discussion of this interesting and highly useful class, of transformations.

APPENDIX

1. The element of area on a locally Euclidean (but otherwise arbitrary) two dimensional surface is given by the well-known formula due to Gauss:[31] $dA = (EG-F^2)^{1/2}\, du\, dv$. The element of density on a surface is given by $dD = D(u, v)\, dA$, where $D(u, v)$ represents the given (expected, probable) value at the point u, v. The general problem hence reduces to one of finding u' and v' as functions of u and v to satisfy

$$\iint_{R'} (E'G' - F'^2)^{1/2} \quad du'dv' = \iint_{R} D(u,v)(EG - F^2)^{1/2} dudv \qquad (1.1)$$

or

$$(E'G' - F'^2)^{1/2}|J| = D(u,v)(EG - F^2)^{1/2} . \qquad (1.2)$$

For a sphere $(EG - F^2)$ is equal to $R^4\cos^2\varphi$ using geographical coordinates φ and λ, or to $R^4\sin^2\rho$, using spherical coordinates ρ and λ . In the present instance the interest is only in plane maps; for a plane, $E'G' - F'^2$ is equal to 1, using rectangular coordinates x and y, or to r^2, using polar coordinates r and θ. The interesting cases will generally be oblique projections, but this requires only a relabeling.

When the Jacobian determinant (J) is written out in full, the following partial differential equations obtain:

$$\frac{\partial x}{\partial \lambda}\frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \lambda} = \pm D(\phi,\lambda)R^2\cos\phi , \qquad (1.3)$$

$$r\left[\frac{\partial r}{\partial \rho}\frac{\partial \theta}{\partial \lambda} - \frac{\partial r}{\partial \lambda}\frac{\partial \theta}{\partial \rho}\right] = \pm D(\rho,\lambda)R^2\sin\rho . \qquad (1.4)$$

2. The difficulty of an explicit solution to **1.3** or **1.4** will depend on the specific form of the density function and the additional conditions applied. As is typical of differential equations, in general there will be an infinitude of particular solutions. Certain simple solutions, however, are immediately apparent. For example, if $\partial x/\partial\varphi = 0$ and $\partial x/\partial\lambda$ is arbitrary, then

$$y = R^2 \int \frac{\pm D(\phi,\lambda)\cos\phi}{\partial x/\partial\lambda}\, d\phi + g(\lambda)\,. \tag{2.1}$$

Or if $\partial y/\partial\lambda = 0$ and $y = f(\varphi)$ is given, then

$$x = R^2 \int \frac{\pm D(\phi,\lambda)\cos\phi}{\partial y/\partial\phi}\, d\lambda + g(\phi)\,. \tag{2.2}$$

In polar coordinates a similar procedure is available. Taking $\partial\theta/\partial\rho = 0$ and a given $\partial\theta/\partial\lambda$ yields

$$r^2 = 2R^2 \int \frac{\pm D(\rho,\lambda)\sin\rho}{\partial\theta/\partial\lambda}\, d\rho + g(\lambda)\,. \tag{2.3}$$

An azimuthal projection is obtained if $\theta = \lambda$, conic projections if $\theta = n\lambda$, etc.. Taking $\partial r/\partial\lambda = 0$, and with $r = f(\rho)$ selected arbitrarily, yields

$$\theta = R^2 \int \frac{\pm D(\rho,\lambda)\sin\rho}{r(\partial r/\partial\rho)}\, d\lambda + g(\rho)\,. \tag{2.4}$$

2. The condition that a map of the sphere be equal-area can he written as

$$|J| / R^2\cos\varphi = 1 \text{ (or constant).} \tag{3.1}$$

Hence it follows immediately that equal-area projections represent the special case D =1 (or constant).

4. Areal distortion (S) is, by definition, the ratio of the element of area on the map to the element of area on die original. In other words,

$$S = \frac{dA'}{dA} = \frac{(E'G' - F'^2)^{1/2}}{(EG - F^2)^{1/2}}\,.$$

From a simple substitution it is seen that the density is the same as the areal distortion (i.e. D = S). In Tissot's notation S = ab, the product of the linear distortion in two orthogonal directions. Knowing this relation, we can obtain the desired transformations by choosing the areal distortion to match exactly the expected or known density distribution.

5. If the density is given by $\cos^{-1}(\rho/2)$ and an azimuthal projection is desired, equation **2.3** yields the stereographic projection. Although such a density is unlikely, this demonstrates the existence of conformal projections within this class of projections. The suggestion is that a conformal version exists among the solutions for many, if not all, non-constant densities. Though the areal distortion on conformal projections is easily calculated, the existence of conformal projections with a given areal distortion involves more subtle considerations, which are not presented hcre.[32]

6. According to Tissot, every non-conformal transformation retains as orthogonal curves one, and only one, pair of curves orthogonal on the original. An interesting question is whether the transformation can be determined so that the lines of latitude and longitude are the lines that remain orthogonal. For densities that depend on only one parameter the condition is readily obtained. For example, if $D = D(\varphi)$ and $\partial x/\partial\lambda = 1$, equation **2.1** yields a cylindrical projection. Korkine's analysis of equal-area projections may be of use in obtaining the general case.[33]

7. Transport costs are often said to increase at a decreasing rate with distance, i.e. $\partial^2 r/\partial\rho^2 < 0$. If $r = f(\rho)$ and a density $D(\rho, \lambda)$ is given, equation **2.4** yields a solution that renders map distances proportional to transport costs and distributes densities evenly (see **8.4**). An even more interesting result would be the simultaneous solution of **1.4** with an arbitrary $r = f(\rho, \lambda)$.

8. A few particular solutions may be of interest. From **2.3** an azimuthal projection for a linearly decreasing density $D = a\rho + b$, $a < 0 < b$, yields

$$r = [2R^2(-a\rho\cos\rho - b\cos\rho + a\sin\rho)]^{1/2} \quad . \qquad (8.1)$$

If the density distribution in Hägerstrand's problem is assumed to be ρ^{-1}**,** the appropriate azimuthal projection is

$$r^2 = 2R^2 \int \frac{\sin\rho}{\rho} d\rho = 2R^2\left(\rho - \frac{\rho^3}{3\cdot 3!} + \frac{\rho^5}{5\cdot 5!} - \frac{\rho^7}{7\cdot 7!} + \frac{\rho^9}{9\cdot 9!}\cdots\right). \qquad (8.2)$$

Additional azimuthal projections for densities equaling $\exp(-\rho)$ or $\exp(-\rho^2/2)$ would appear to be of geographic interest, and are relatively easily obtained.

From **2.4** one obtains an equidistant version with $r = R\rho$ and $D = \pi - \rho$:

$$\theta = (-1 + \pi/\rho)\,\lambda \sin\rho. \qquad (8.3)$$

Also from **2.4** but with $r = R(\rho)^{1/2}$, $D = \pi - \rho$, one has

$$\theta = 2\lambda(\pi - \rho)\sin\rho + g(\rho). \qquad (8.4)$$

In all these instances it is necessary to examine the resulting transformation for one-to-oneness. Choice of the constants of integration may be of importance. In some instances the substitution of difference equations for the differential equations may be appropriate. The author has calculated further special cases, which will be made available to interested parties.

9.It is suggested that these projections be referred to by their mathematical name; that is, as transformations of surface integrals.

NOTES

The Geographical Review, LIII, 1 (1963), pp. 59-78.

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[1]Edgar Kant: Umland Studies and Sector Analysis, in Studies in Rural-Urban interaction, *Lund Studies in Geography,* Ser. B, Human Geography, No. 3, 1951, pp. 3-13; reference on p. 5. Italics are Kant's.

[2]Johann H. von Thünen: Der isolierte Staat in Beziehung auf Landwirtschaft und National-Ökonomie (Hamburg, 1826).

[3]Carlisle W. Baskin: A Critique and Translation of Walter Christaller's *Die zentralen Orte in Siiddeutschland* (unpublished Ph.D. dissertation, Department of Economics, University of Virginia, 1957).

[4]Harold and Margaret Sprout: Environmental Factors in the Study of International Politics, *J. of Conflict Resolution*, Vol. 1, 1957, pp. 309-328.

[5]Chauncy D. Harris: The Market as a Factor in the Localization of Industry in the United States, *Annals,* Assn. of Amer. Geogrs., Vol. 44, 1954, pp. 3 15-348. See also Chauncy D. Harris and George B. McDowell: Distorted Maps, A Teaching Device, *J. of Geogr.,* Vol. 54, 1955, pp. 286—289.

[6]Erwin Raisz: The Rectangular Statistical Cartogram, *Geog. Rev.,* Vol. 24, 1934, pp. 292-296, Fig. 2 (p. 293).

[7]Edgar M. Hoover: The Location of Economic Activity (New York, Toronto, London, 1948), Fig. 5.6 (p. 88).

[8]Hans W. Weigert and others: Principles of Political Geography (New York, 1957), Fig. 9.2 (p. 296).

[9]W. S. Woytinsky and E. S. Woytinsky: World Population and Production (New York, 1953), pp. Lxix-lxxii and 42-43, and *passim.*

[10]Erich W. Zimmermann: World Resources and Industries (rev, edit., New York, 1951), p. 97. Another cartogram can be seen in David Greenhood: Down to Earth: Mapping for Everybody ([rev. edit.] New York, 1951), p. 236. The Library of Congress map collection also contains a large number of' maps of this type.

[11]Torsten Hägerstrand: Migration and Area, in**:** Migration in Sweden, *Lund Studies in Geography***,** Ser. B, Human Geography, No. 13, 1957, pp. 27-158; reference is on p. 73. Italics are Hägerstrand's.

[12]Cf. Arthur H. Robinson: Elements of Cartography (2nd edit.; New York and London, 1960), Figs. 9.16 (p. 169) and 9.17 (p. 170)**.**

[13]Hägerstrand, *op. cit.* 11 above], p. 74. The reference is to P.-H. Chombart de Lauwe and others: Paris et l'agglomération parisienne (2 vols.; Paris, 1952).

[14]For further details on this procedure see W. R. Tobler: Map Transformations of Geographic Space (unpublished Ph.D. dissertation, University of Washington, 1961), pp. 114-117.

[15]F. W. Michels: Drie nieuwe kaartvormen, *Tjjdschr. Kon. Nederl. Aardrijksk. Genootschap,* Ser. 2, Vol. 76, 1959, pp. 203-209. See also D. M. Desoutter: Projection by Introspection, *Aeronautics,* Vol. 40, 1959**,** pp. 42-44.

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17[Leslie Y. Dameron:] Terminal Area Charts for jet Aircraft, *Military Engineer,* Vol. 52, 1960, p**.** 227,

[18]Harris, *op. cit.*[see footnote 5 above], p. 338.

[19]See C. B. P. Brooks and N. Carruthers: Handbook of Statistical Methods in Meteorology, M.O. 538*,* London, 1953, pp. 161-165; or Calvin F. Schmid and Earle H. MacCannell: Basic Problems, Techniques, and Theory of Isopleth Mapping, *J. Amer. Statist. Assn.,* Vol. 50, 1955, pp. 220-239.

[20]E. M. Horwood: A Three-Dimensional Calculus Model of Urban Settlement (paper presented at the Regional Science Association Symposium, Stockholm, August, 1960).

[21]Erwin Raisz: Rectangular Statistical Cartograms of the World, *Journ. Of Geogr.***,** Vol. 35, 1936, pp. 8-10; idem**,** The Rectangular Statistical Cartogram [see footnote 6 above].

[22]The plotting can be conceptual, or it can be internal in a digital computer, and need not actually be performed.

[23]Edgar S. Dunn, Jr.: The Location of Agricultural Production (Gainesville, Fla., 1954), pp. 67-69.

[24]A more extensive discussion of this topic can be found in Tobler, *op. cit.* [see footnote 14 above], pp. 78-141.

[25]William Alonso: A Model of the Urban Land Market: Locations and Densities of

Dwellings and Businesses (unpublished Ph.D. dissertation, University of Pennsylvania, 1960).

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[27]Arthur Getis: A Theoretical and Empirical Inquiry into the Spatial Structure of Retail Activities (unpublished Ph.D. dissertation, University of Washington, 1961), pp. 89-102.

[28]Lewis P. Richardson: The Problem of Contiguity (an appendix to his "The Statistics of Deadly Quarrels" [Pittsburgh, Chicago, and London,1960]), *General Systems*, Vol. 6, 1962, pp. 139-187.

[29]Walter Isard: Location and Space Economy ([Cambridge and] New York, 1956), Figs. 52 (p.272), 53 (p.277), and 54 (p. 279).

[30]Marcello Vidale: A Graphical Solution to the Transportation Problem, *Operations Research,* Vol. 4, 1956, pp. 193-203.

[31]See, for example, Dirk J. Struik: Lectures on Classical Differential Geometry (Reading, Mass., 1950), or any other text on differential geometry. Einstein's more convenient notation is not employed in cartography.

[32]See, for example, Richardson, *op. cit.* [see footnote 28 above], equation 4.54 (p. 158).

[33]A. Korkine: Sur les cartes géographiques, *Mathematische Annalen,,* Vol. 35, 1890, pp. 588-604. Also note the similarity to equations derived by the Russian Urmaev, as discussed in D. H. Maling: A Review of Some Russian Map Projections, *Empire Survey Rev*., Vol. 15, 1959-1960, pp. 203-215, 255-266, and 294-303 references on pp. 210-213.

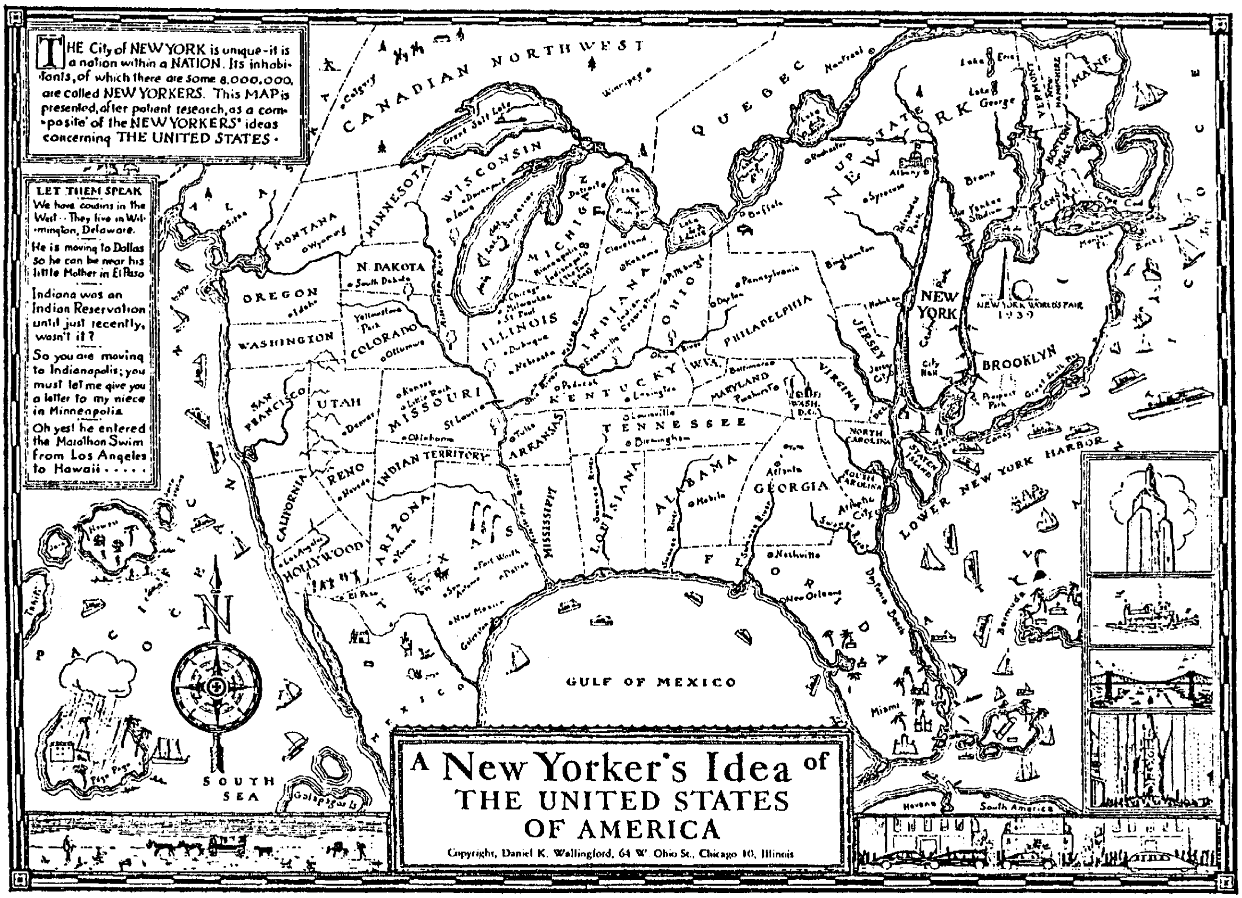

FIG. 1—This map is interesting viewed as a projection; but the present emphasis is on distortion of area. (Courtesy of Daniel K. Wallingford.)

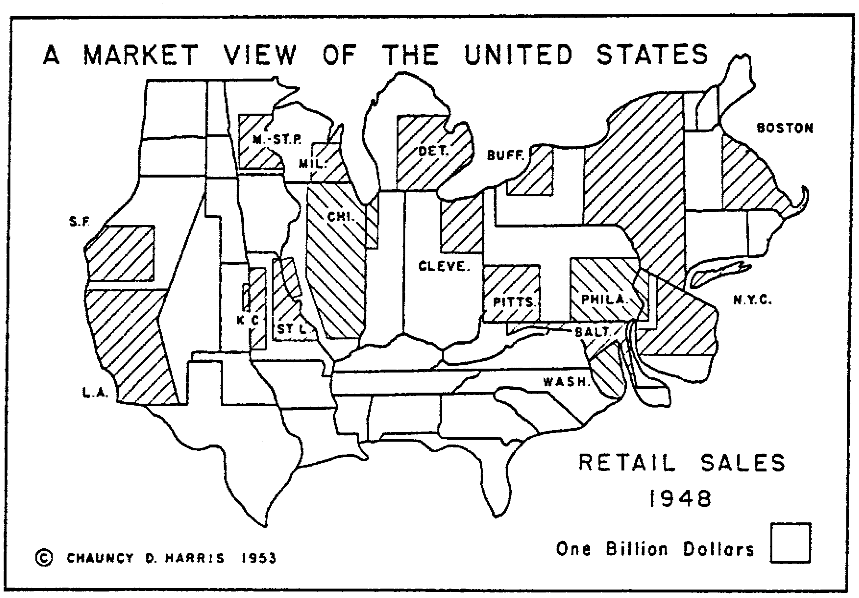

FIG. 2—Sizes of states proportional to retail sales, 1948. Major cities are shaded. From Harris, The Market as a Factor in the Localization of Industry [see text footnote 5 below], p. 320. (Courtesy of Chauncy D. Harris.)

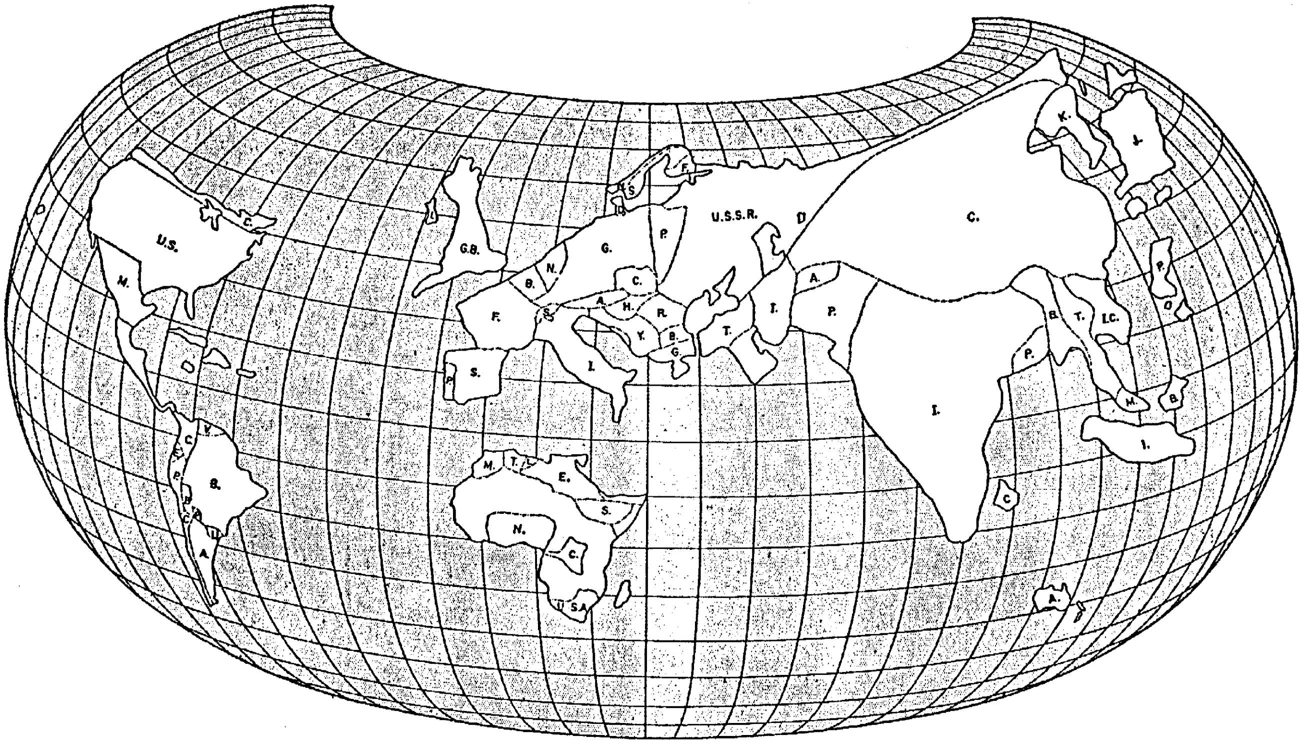

FIG. 3—Continents and selected countries on the scale of their population. The background of latitude and longitude is spurious. From Woytinsky and Woytinsky, World Population and Production [see text footnote 9 below], p. 42. (Courtesy of the Twentieth Century Fund.)

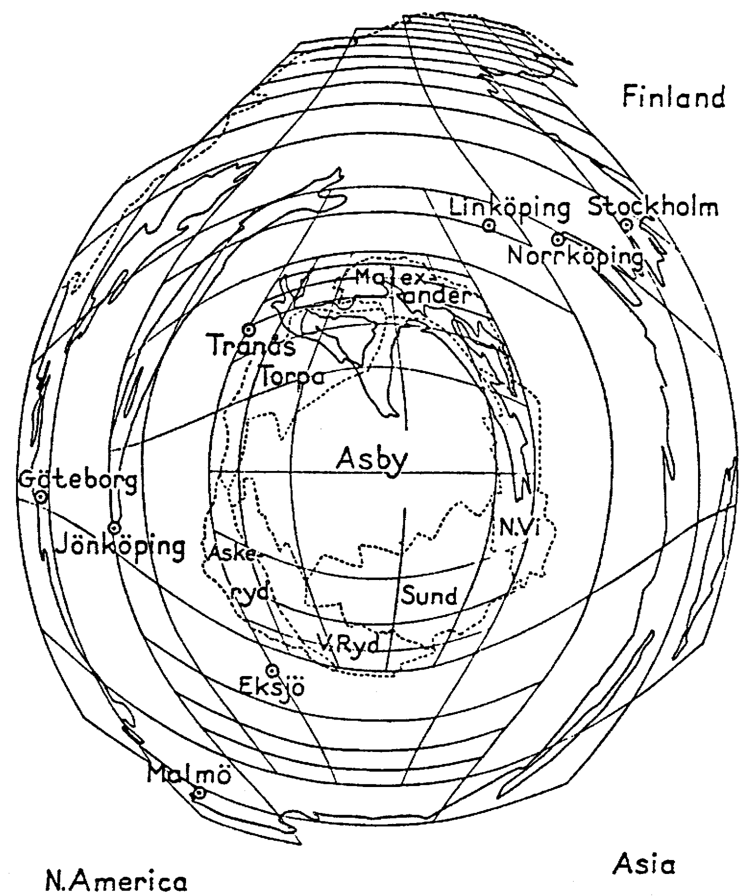

FIG. 4—Hägerstrand's logarithmic map. The ~~numbers~~ and grid lines refer to the Swedish plane coordinate system. From Hägerstrand, Migration and Area [see text footnote 11 above], p. 73 (Courtesy of Torsten Hägerstrand.)

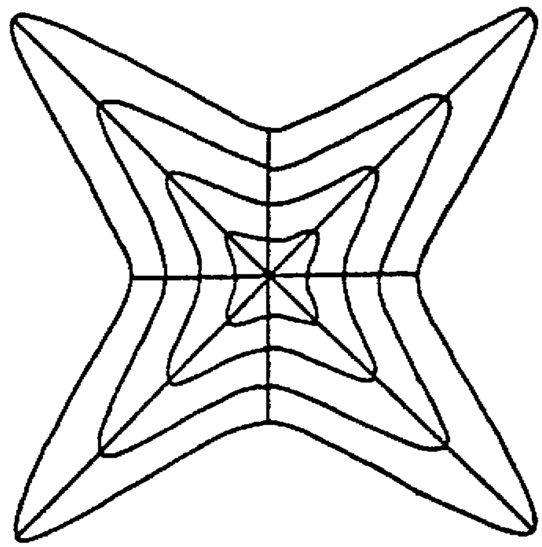

FIG. 5—This illustration can be considered as either (a) isolines of population density or (*b*) polar co-ordinates after a transformation.

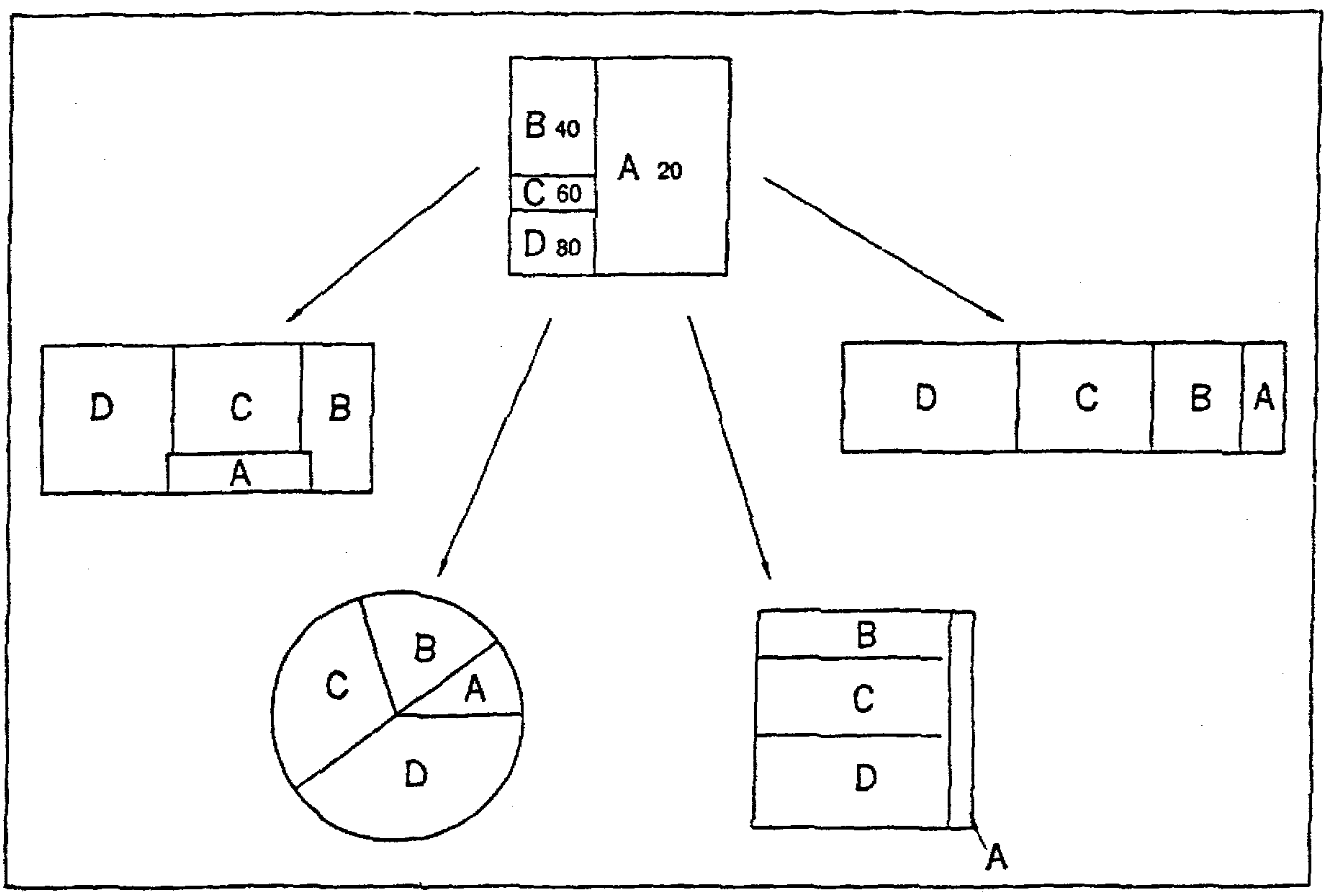

FIG. 6—Sample transformations of a unit square with different amounts of a phenomenon in different portions illustrating several of the possible solutions.